# Continuous phase transitions: Subgroups of index $N$

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We demonstrate that it is not possible to extend Landau's subgroup of index-three theorem for continuous phase transitions to subgroups of index $N$ with $N \neq 3$.

The Landau conjecture,[1] or subgroup of index-three theorem, states that all phase transitions between phases of an arbitrary symmetry $G$ and all subgroups $H$ of index three in $G$ are not continuous. This was shown to be valid for the special case of cubic to tetragonal phase transitions by Anderson and Blount.[2] For phase transitions characterized by a deformation of the unit cell, a proof was given by Boccara.[3] General proofs of the subgroup of index-three theorem have been given by Meisel, Gray, and Brown[4] and Brown and Meisel.[5] These general proofs show that all irreducible representations associated with phase transitions between phases of an arbitrary symmetry group $G$ and subgroup $H$ of index three in $G$ do not satisfy the Landau criterion[1] for continuous phase transitions.

One may be tempted to believe that it would be possible to extend this theorem in some simple way to values of $N \neq 3$. We now show that the subgroup of index-three theorem cannot be extended to a subgroup of index-$N$ theorem with $N \neq 3$. This is done by constructing a group $G$ and irreducible representation of $G$ associated with a phase transition between $G$ and subgroup $H$ of index $N$ in $G$ which satisfies the subduction,[6] chain subduction,[7] and Landau criteria[1] for continuous phase transitions. As the case of $N=2$ is well known,[1] we limit our discussion to the case of $N>3$.

Consider the group $G=C_N \times H$, the direct product of a cyclic group of order $N$ generated by an element denoted by $c$, $c^N=E$, and an arbitrary group $H$. The irreducible representations of $G$ are direct products of the irreducible representations of the groups $C_N$ and $H$. The irreducible representations of $C_N$ are one dimensional, will be denoted by $\Gamma^\alpha$, $\alpha=0,1,2,\ldots,N-1$, and are defined by

$$\Gamma^\alpha(c^\beta)=\omega^{\alpha\beta} \ , \tag{1}$$

where $\omega=e^{2\pi i/N}$. The irreducible representations of $H$ will be denoted by $D_H^i$.

We consider the phase transition between $G=C_N \times H$ and subgroup $H$ of index $N$ in $G$ and the associated physically irreducible representation

$$D_G=\begin{pmatrix}\Gamma^1 & \\ & \Gamma^{N-1}\end{pmatrix}\times D_H^1 \ . \tag{2}$$

Since $\Gamma^1$ is complex, we have coupled $\Gamma^1$ with $\Gamma^{1*}=\Gamma^{N-1}$ to form a single physically irreducible representation.[8] $D_H^1$ is the identity representation of $H$. We now show that this representation $D_G$ associated with this phase transition satisfies the subduction, chain subduction, and Landau criteria for a continuous phase transition.

(a) Subduction criterion: Let $M_H$ denote the number of times the identity representation $D_H^1$ of $H$ is contained in the representation $D_G$ subduced on $H$, i.e., restricted to the elements of the subgroup $H$ of $G$. A representation $D_G$ associated with a phase transition between phases of symmetry $G$ and $H$ satisfies the subduction criterion if $M_H>0$. The representation $D_G$ given in Eq. (2), with $M_H=2$, satisfies the subduction criterion.

(b) Chain subduction criterion: Let $G$ contain a proper subgroup $K$ which in turn contains the subgroup $H$. A representation $D_G$ associated with a phase transition between phases of symmetry $G$ and $H$ satisfies the chain subduction criterion if $M_K < M_H$.

For the representation $D_G$, Eq. (2), of $G=C_N\times H$, $M_H=2$. Proper subgroups $K$ of $G$ are of the form $C_{N/Z}\times H$ where $Z$ is a factor of $N$, and $Z\neq 1$ or $N$. If $N$ is prime, there are no such subgroups, and $D_G$ satisfies the chain subduction criterion. If $N$ is not prime, then using Eqs. (1) and (2), $M_K$ is calculated from

$$M_K=\frac{1}{N/Z}\sum_{\gamma=0}^{(N/Z)-1}\{\exp[2\pi i\gamma/(N/Z)] +\exp[-2\pi i\gamma/(N/Z)]\} \ . \tag{3}$$

Since $N/Z\neq 1$, $M_K=0$, and $D_G$ satisfies the chain subduction criterion.

(c) Landau criterion: A representation $D_G$ satisfies the Landau criterion if the symmetrized cube of the representation does not contain the identity representation of $G$. We show that $D_G$, Eq. (2), satisfies the Landau criterion by showing that the cube of $D_G$ does not contain the identity representation of $G$. The number of times the identity representation of $G$ is contained in the cube of $D_G$ is proportional to

$$\sum_{\beta=0}^{N-1}(\omega^{\beta}+\omega^{-\beta})^3=2\,\mathrm{Re}\left[\sum_{\beta=0}^{N-1}(\omega^{3\beta}+\omega^{\beta})\right] . \quad (4)$$

The $N$ $N$th roots of unity satisfy

$$\sum_{\beta=0}^{N-1}\omega^{\nu\beta}=\begin{cases}0 \text{ if } N \text{ is not a factor of } \nu\\ N \text{ if } N \text{ is a factor of } \nu\end{cases} . \quad (5)$$

Since $N>3$, the summation on the right-hand side of Eq. (4) is zero and the identity representation of $G$ is not contained in the cube of $D_G$. $D_G$ then satisfies the Landau criterion.

The representation $D_G$, Eq. (2), associated with a phase transition between phases of symmetry $G=C_N \times H$ and subgroup $H$ of index $N$ in $G$ therefore satisfies the subduction, chain subduction, and Landau criteria for continuous phase transitions. Consequently, this is an example of an allowed phase transition between a phase of symmetry $G$ and subgroup $H$ of index $N>3$ in $G$. Thus the Landau subgroup of index-three theorem can not be extended to subgroups of index $N \neq 3$.

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[1]L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1958), Chap. 14.
[2]P. W. Anderson and E. I. Blount, Phys. Rev. Lett. 14, 217 (1966).
[3]N. Boccara, Ann. Phys. (N.Y.) 47, 40 (1968).
[4]L. V. Meisel, D. M. Gray, and E. Brown, J. Math. Phys. (N.Y.) 16, 2520 (1975).
[5]E. Brown and L. V. Meisel, Phys. Rev. B 13, 213 (1976).
[6]J. L. Birman, Phys. Rev. Lett. 17, 1216 (1966).
[7]F. E. Goldrich and J. L. Birman, Phys. Rev. 167, 528 (1968).
[8]G. Ya. Lyubarskii, *The Application of Group Theory in Physics* (Pergamon, Oxford, 1960).