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Symmetry and phase transitions in decagonal quasicrystals

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Abstract. The possible non-crystallographic point group of the decagonal quasicrystal phase of Al–Mn alloys has been shown by Bendersky to be either D_{10h} or C_{10h}. For the physically irreducible representations of these groups, we derive the Clebsch–Gordan products, extended integrity bases, stability spaces and tensorial covariants. The point groups which can arise in phase transitions are determined along with corresponding tensorial parameters which could drive the transition. It is shown that equilibrium tensorial properties whose components transform as the components of the electrogyration or elasto-optic tensors can distinguish between the D_{10h} and C_{10h} point group symmetry of the decagonal phase.

1. Introduction

The decagonal or *T*-phase quasicrystal is a quasicrystal with one-dimensional translational symmetry and tenfold rotational symmetry. Bendersky (1985, 1986) has shown that the non-crystallographic point group symmetry of this quasicrystal is either $D_{10h}(10/mmm)$ or $C_{10h}(10/m)$.

In this paper we examine the group theoretical properties of the physically irreducible representations (PIR) of the point groups D_{10h} and C_{10h} and their implications for phase transitions and tensorial properties of quasicrystals with such point group symmetries. In § 2 we define the PIR of D_{10h} and C_{10h}, the Clebsch–Gordan series, Clebsch–Gordan products and the extended integrity bases for these point groups. The subgroups of D_{10h} and C_{10h}, and the stability spaces of their PIR, are derived in § 3. We also determine in § 3 the possible subgroup symmetries which can arise during a phase transition. In § 4 we derive tensorial covariants which can be transition parameters and discuss the tensorial invariants which can distinguish between the D_{10h} and C_{10h} point group symmetry.

2. Physically irreducible representations

Generators of a set of physically irreducible representations (PIR) of the point groups D_{10} and C_{10} are given in table 1. The non-standard indexing of the PIR of the point

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Table 1. Physically irreducible representations (PIR) of the point groups C_{10} and D_{10}, $s = \sin 2\pi/20$ and $c = \cos 2\pi/20$.

Point group $D_{10}(10_z 2_x 2_y)$			Point group $C_{10}(10_z)$	
PIR	10_z	2_x	PIR	10_z
D_1	1	1	D_1	1
D_2	1	−1	D_2	−1
D_3	−1	1		
D_4	−1	−1		
D_5	$\begin{pmatrix} s & -c \\ c & s \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	D_5	$\begin{pmatrix} s & -c \\ c & s \end{pmatrix}$
D_6	$\begin{pmatrix} -c & -s \\ s & -c \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	D_6	$\begin{pmatrix} -c & -s \\ s & -c \end{pmatrix}$
D_7	$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	D_7	$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$
D_8	$\begin{pmatrix} -s & -c \\ c & -s \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	D_8	$\begin{pmatrix} -s & -c \\ c & -s \end{pmatrix}$

group C_{10} has been chosen to explicitly show relationships between the PIR of D_{10} and C_{10}. The PIR D_i, $i=1, 2$, and D_i, $i=3, 4$, of the point group D_{10} subduced onto the point group C_{10} are, respectively, the PIR D_i, $i=1, 2$, of the point group C_{10}. The PIR D_i, $i=5, 6, 7, 8$, of D_{10} subduced onto C_{10} are, respectively, the PIR D_i, $i=5, 6, 7, 8$, of the point group C_{10}. The PIR of $D_{10h}=D_{10}\times\bar{1}$ and $C_{10h}=C_{10}\times\bar{1}$ are denoted, as is customary, by the symbols D_i^+ and D_i^-.

The Clebsch-Gordan series for the PIR of the point groups D_{10} and C_{10} are given in table 2. At the intersection of the ith row and the jth column are the indices k of

Table 2. Clebsch-Gordan series for the PIR of the point groups C_{10} and D_{10}.

D_{10}	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2		1	4	3	5	6	7	8
3			1	2	8	7	6	5
4				1	8	7	6	5
5					1+2+6	5+6	7+8	3+4+7
6						1+2+5	3+4+8	7+8
7							1+2+5	5+6
8								1+2+6

C_{10}	1	2	5	6	7	8
1	1	2	5	6	7	8
2		1	8	7	6	5
5			1+1+6	5+6	7+8	2+2+7
6				1+1+5	2+2+8	7+8
7					1+1+5	5+6
8						1+1+6

the PIR which appear in the reduced form of the direct product $D_i \times D_j$:

$$D_i \times D_j = \sum \oplus D_k. \tag{1}$$

If a specific PIR appears more than once on the right-hand side of equation (1), the corresponding index k is repeated in table 2. This table also gives the Clebsch-Gordan series for the point groups D_{10h} and C_{10h}. We have

$$D_i^+ \times D_j^+ = D_i^- \times D_j^- = \sum \oplus D_k^+ \tag{2a}$$

$$D_i^+ \times D_j^- = D_i^- \times D_j^- = \sum \oplus D_k^- \tag{2b}$$

and indices k are again found at the intersection of the ith row and jth column of table 2.

Table 3. Clebsch-Gordan products for the PIR of the point group D_{10}.

PIR	Basis functions
D_1	X_2^2, X_3^2, X_4^2, $X_5^2+Y_5^2$, $X_6^2+X_6^2$, $X_7^2+Y_7^2$, $X_8^2+Y_8^2$
D_2	X_3X_4, $X_5Y_5-Y_5X_5$, $X_6Y_6-Y_6X_6$, $X_7Y_7-Y_7X_7$, $X_8Y_8-Y_8X_8$
D_3	X_2X_4, $X_5X_8-Y_5Y_8$, $X_6X_7-Y_6Y_7$
D_4	X_2X_3, $X_5Y_8+Y_5X_8$, $X_6Y_7+Y_6X_7$
D_5	$X_2(Y_5, -X_5)$, $X_3(X_8, -Y_8)$, $X_4(Y_8, X_8)$, $(X_7^2-Y_7^2, X_7Y_7+Y_7X_7)$, $(X_6^2-Y_6^2, -X_6Y_6-Y_6X_6)$, $(X_5X_6+Y_5Y_6, X_5Y_6-Y_5X_6)$, $(X_7X_8+Y_7Y_8, X_7Y_8-Y_7X_8)$
D_6	$X_2(Y_6, -X_6)$, $X_3(X_7, -Y_7)$, $X_4(Y_7, X_7)$, $(X_5^2-Y_5^2, X_5Y_5+Y_5X_5)$, $(X_8^2-Y_8^2, -X_8Y_8-Y_8X_8)$, $(X_5X_6-Y_5Y_6, -X_5Y_6-Y_5X_6)$, $(X_7X_8-Y_7Y_8, X_7Y_8+Y_7X_8)$
D_7	$X_2(Y_7, -X_7)$, $X_3(X_6, -Y_6)$, $X_4(Y_6, X_6)$, $(X_5X_7+Y_5Y_7, -X_5Y_7+Y_5X_7)$, $(X_6X_8+Y_6Y_8, -X_6Y_8+Y_6X_8)$, $(X_5X_8+Y_5Y_8, X_5Y_8-Y_5X_8)$
D_8	$X_2(Y_8, -X_8)$, $X_3(X_5, -Y_5)$, $X_4(Y_5, X_5)$, $(X_5X_7-Y_5Y_7, X_5Y_7+Y_5X_7)$, $(X_6X_7+Y_6Y_7, -X_6Y_7-Y_6X_7)$, $(X_6X_8-Y_6Y_8, -X_6Y_8-Y_6X_8)$

We shall use the following notation for the basis functions of the PIR defined in table 1. For the one-dimensional PIR D_i, $i = 1, 2, 3, 4$, of the point group D_{10} and D_i, $i = 1, 2$, of the point group C_{10}, we denote the basis functions by X_i, $i = 1, 2, 3, 4$. For the two-dimensional PIR D_i, $i = 5, 6, 7, 8$, of both point groups D_{10} and C_{10} we denote the basis functions as (X_i, Y_i), $i = 5, 6, 7, 8$. For the groups D_{10h} and C_{10h}, one includes a superscript '+' or '−' in the above notation for the basis functions of PIR with the same superscript notation.

The linear combinations of products of basis functions of PIR D_i and D_j which are basis functions of the PIR D_k appearing on the right-hand side of (1) are known as Clebsch-Gordan products (Kopsky 1976). The Clebsch-Gordan products for the PIR of the point groups D_{10} and C_{10} are given, respectively, in tables 3 and 4. These same tables represent the Clebsch-Gordan products for the PIR of the point groups D_{10h} and C_{10h}. For PIR D_k^+ appearing on the right-hand side of (2a), one includes in tables 3 and 4 the superscript '+' or '−' on all basis functions. For PIR D_k^- (see (2b)), one includes in tables 3 and 4 the superscript '+' on the first basis function and the superscript '−' on the second basis function, or vice versa, in each term of the Clebsch-Gordan products.

An extended integrity basis of a polynomial algebra in a set of variables on which a finite group operates includes the ordinary integrity basis of invariants and the linear integrity basis of covariants (Kopsky 1975, 1979a, Patera *et al* 1978). The latter are

Table 4. Clebsch-Gordan products for the PIR of the point group C_{10}.

PIR	Basic functions
D_1	$X_2^2, X_5^2+Y_5^2, X_6^2+Y_6^2, X_7^2+Y_7^2, X_8^2+Y_8^2, X_5Y_5-Y_5X_5, X_6Y_6-Y_6X_6, X_7Y_7-Y_7X_7, X_8Y_8-Y_8X_8$
D_2	$X_5X_8-Y_5Y_8, X_6X_7-Y_6Y_7, X_5Y_8+Y_5X_8, X_6Y_7+Y_6X_7$
D_5	$X_2(X_8, -Y_8), (X_7^2-Y_7^2, X_7Y_7+Y_7X_7), (X_6^2-Y_6^2, -X_6Y_6-Y_6X_6), (X_5X_6+Y_5Y_6, X_5Y_6-Y_5X_6), (X_7X_8+Y_7Y_8, X_7Y_8-Y_7X_8)$
D_6	$X_2(X_7, -Y_7), (X_5^2-Y_5^2, X_5Y_5+Y_5X_5), (X_8^2-Y_8^2, -X_8Y_8-Y_8X_8), (X_5X_6-Y_5Y_6, -X_5Y_6-Y_5X_6), (X_7X_8-Y_7Y_8, X_7Y_8+Y_7X_8)$
D_7	$X_2(X_6, -Y_6), (X_5X_7+Y_5Y_7, -X_5Y_7+Y_5X_7), (X_5X_8+Y_5Y_8, X_5Y_8-Y_5X_8), (X_6X_8+Y_6Y_8, -X_6Y_8+Y_6X_8)$
D_8	$X_2(X_5, -Y_5), (X_5X_7-Y_5Y_7, X_5Y_7+Y_5X_7), (X_6X_7+Y_6Y_7, -X_6Y_7+Y_6X_7), (X_6X_8-Y_6Y_8, -X_6Y_8-Y_6X_8)$

defined as sets of covariants of a given type such that any other covariant of this type is expressible as a linear combination of the basic ones with invariants as coefficients of the combination. With the aid of the Clebsch-Gordan products we have derived the extended integrity basis for the point groups D_{10h} and C_{10h}. In table 5 we give the extended integrity basis for the point group D_{10h}, i.e. for the polynomial algebras where the set of variables are the basis functions of the PIR of the point group D_{10h}. The extended integrity basis for the point group C_{10h} is given in table 6, where that part of the table not explicitly given is identical with the corresponding part in table 5. In both tables 5 and 6 we have used the shorthand notations P_i and Q_i, $i = 1, 2, \ldots, 10$, for polynomials which are defined in table 7. For typographical simplicity the basis functions have been entered with neither subscript nor superscript. The subscript and superscript of all basis functions in a specific row of tables 5 and 6 are that of the PIR indexing that row.

3. Phase transitions

In this section we determine the stability spaces of the PIR of the point groups D_{10h} and C_{10h}, and consequently the possible subgroup symmetries which can arise via a phase transition. In figure 1 we show the coordinate system used and the axes of the twofold rotations denoted by $2_x^{(j)}$ and $2_y^{(j)}$, $j = 1, 2, \ldots, 5$. In table 8 we list the elements of the point group D_{10h} and of the subgroups of D_{10h}. The elements of the point group C_{10h} and of the subgroups of C_{10h} are also found in this table. A superscript 'j' in the symbol of a subgroup in table 8, e.g. $D_{2h}^{(j)}$, signifies that this symbol denotes five subgroups $D_{2h}^{(j)}$, $j = 1, 2, \ldots, 5$. Figures 2 and 3 give, diagrammatically, the relationships between the point groups D_{10h} and C_{10h} and their respective subgroups.

The stability space of a PIR D_i of a group G_0 with respect to a subgroup G of G_0 is that subspace of the space spanned by the basis functions of the PIR D_i, all vectors of which are invariant under G (Kopsky 1983). In table 9 we list the stability spaces of all PIR of the point group C_{10h} with respect to all subgroups of C_{10h}. The one-dimensional stability spaces of one-dimensional PIR are denoted by the symbol of the corresponding basis function. Two-dimensional stability spaces of two-dimensional PIR are denoted by the symbol of the corresponding PIR.

Table 5. Extended integrity basis for the point group D_{10h}.

	X_1^+	X_2^+	X_3^+	X_4^+	(X_5^+, Y_5^+)	(X_6^+, Y_6^+)	(X_7^+, Y_7^+)	(X_8^+, Y_8^+)	X_1^-	X_2	X_3^-	X_4	(X_5^-, Y_5)	(X_6^-, Y_6)	(X_7, Y_7)	(X_8, Y_8)
D_1^+	X															
D_2^+	X^2	X														
D_3^+	X^2		X													
D_4^+	X^2			X												
D_5^+	X^2+Y^2	Q_5			(X, Y)	(P_2, Q_2)										
	P_5				$(P_4, -Q_4)$	$(P_3, -Q_3)$										
D_6^+	X^2+Y^2	Q_5			$(P_2, -Q_2)$	(X, Y)										
	P_5				(P_3, Q_3)	$(P_4, -Q_4)$										
D_7^+	X^2+Y^2	Q_{10}	P_5	Q_5	(P_2, Q_2)	(P_4, Q_4)	(X, Y)	(P_3, Q_3)								
	P_{10}				$(P_8, -Q_8)$	$(P_6, -Q_6)$	$(P_9, -Q_9)$	$(P_7, -Q_7)$								
D_8^+	X^2+Y^2	Q_{10}	P_5	Q_5	(P_4, Q_4)	$(P_2, -Q_2)$	$(P_3, -Q_3)$	(X, Y)								
	P_{10}				$(P_6, -Q_6)$	(P_8, Q_8)	(P_7, Q_7)	$(P_9, -Q_9)$								
D_1	X^2								X							
D_2	X^2									X						
D_3^-	X^2										X					
D_4	X^2											X				
D_5^-	X^2+Y^2	Q_{10}			$(P_4, -Q_4)$	(P_2, Q_2)			P_5	Q_5			(X, Y)	$(P_3, -Q_3)$		
	P_{10}				(P_6, Q_6)	$(P_8, -Q_8)$							$(P_9, -Q_9)$	(P_7, Q_7)		
D_6	X^2+Y^2	Q_{10}			$(P_2, -Q_2)$	$(P_4, -Q_4)$			P_5	Q_5			(P_3, Q_3)	(X, Y)		
	P_{10}				(P_8, Q_8)	(P_6, Q_6)							$(P_7, -Q_7)$	$(P_9, -Q_9)$		
D_7	X^2+Y^2	Q_{10}			(P_2, Q_2)	(P_4, Q_4)					P_5	Q_5			(X, Y)	(P_3, Q_3)
	P_{10}				$(P_8, -Q_8)$	$(P_6, -Q_6)$									$(P_9, -Q_9)$	$(P_7, -Q_7)$
D_8^-	X^2+Y^2	Q_{10}			(P_4, Q_4)	$(P_2, -Q_2)$					P_5	Q_5			$(P_3, -Q_3)$	(X, Y)
	P_{10}				$(P_6, -Q_6)$	(P_8, Q_8)									(P_7, Q_7)	$(P_9, -Q_9)$

Table 6. Extended integrity basis for the point group C_{10h}. The part of this table not explicitly given is identical with the corresponding part of table 5.

	X_1^+	X_2^+	X_1^-	X_2^-
D_1^+	X			
D_2^+	X^2	X		
D_5^+	X^2+Y^2, P_5, Q_5			
D_6^+	X^2+Y^2, P_5, Q_5			
D_7^+	X^2+Y^2, P_{10}, Q_{10}	P_5, Q_5		
D_8^+	X^2+Y^2, P_{10}, Q_{10}	P_5, Q_5		
D_1^-	X^2		X	
D_2^-	X^2			X
D_5^-	X^2+Y^2, P_{10}, Q_{10}		P_5, Q_5	
D_6^-	X^2+Y^2, P_{10}, Q_{10}		P_5, Q_5	
D_7^-	X^2+Y^2, P_{10}, Q_{10}			P_5, Q_5
D_8^-	X^2+Y^2, P_{10}, Q_{10}			P_5, Q_5

Table 7. Polynomial abbreviations used in tables 5 and 6.

$P_1 = X$
$P_2 = X^2 - Y^2$
$P_3 = X^3 - 3XY^2$
$P_4 = X^4 - 6X^2Y^2 + Y^4$
$P_5 = X^5 - 10X^3Y^2 + 5XY^4$
$P_6 = X^6 - 15X^4Y^2 + 15X^2Y^4 - Y^6$
$P_7 = X^7 - 21X^5Y^2 + 35X^3Y^4 - 7XY^6$
$P_8 = X^8 - 28X^6Y^2 + 70X^4Y^4 - 28X^2Y^6 + Y^8$
$P_9 = X^9 - 36X^7Y^2 + 126X^5Y^4 - 84X^3Y^6 + 9XY^8$
$P_{10} = X^{10} - 45X^8Y^2 + 210X^6Y^4 - 210X^4Y^6 + 45X^2Y^8 - Y^{10}$

$Q_1 = Y$
$Q_2 = 2XY$
$Q_3 = 3X^2Y - Y^3$
$Q_4 = 4XY(X^2 - Y^2)$
$Q_5 = 5X^4Y - 10X^2Y^3 + Y^5$
$Q_6 = 6X^5Y - 20X^3Y^3 + 6XY^5$
$Q_7 = 7X^6Y - 35X^4Y^3 + 21X^2Y^5 - Y^7$
$Q_8 = 8X^7Y - 56X^5Y^3 + 56X^3Y^5 - 8XY^7$
$Q_9 = 9X^8Y - 84X^6Y^3 + 126X^4Y^5 - 36X^2Y^7 + Y^9$
$Q_{10} = 10X^9Y - 120X^7Y^3 + 252X^5Y^5 - 120X^3Y^7 + Y^{10}$

The stability spaces of all PIR of the point group D_{10h} are given in table 10 with respect to all subgroups of D_{10h}. The one-dimensional stability spaces of one-dimensional PIR are again denoted by the corresponding basis functions and two-dimensional stability spaces of two-dimensional PIR by the symbol of the corresponding PIR. The following notation is introduced to denote the one-dimensional stability spaces of two-dimensional PIR. Let $e_x^{(k)}$ and $e_y^{(k)}$, $k = 1, 2, \ldots, 5$, denote directions in

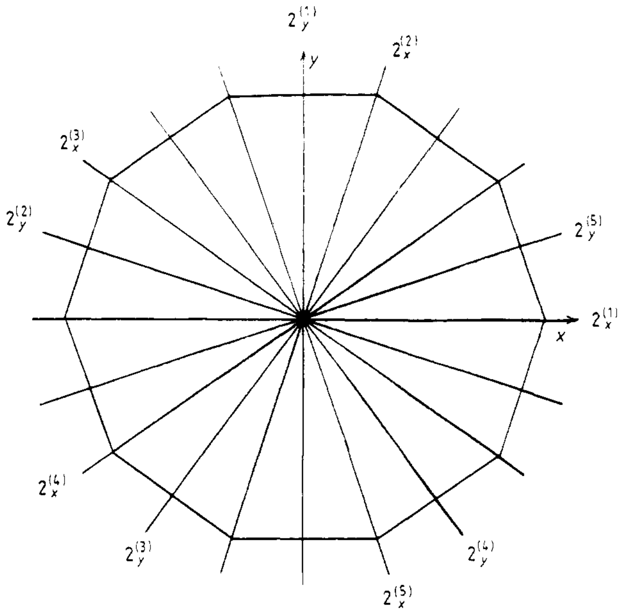

Figure 1. C_{10h} and D_{10h} coordinate system. The z axis is perpendicular to the plane of the figure.

the space spanned by the basis functions of a two-dimensional PIR. The relative orientation of these directions is shown in figure 4. We denote the one-dimensional stability spaces of two-dimensional PIR by $E_{ix}^{+(k)}$, $E_{ix}^{-(k)}$, $E_{iy}^{+(k)}$ and $E_{iy}^{-(k)}$, $i = 5, 6, 7, 8$ and $k = 1, 2, \ldots, 5$. The symbol $E_{ix}^{+(k)}$, for example, denotes the one-dimensional stability space in the space spanned by the basis functions of the PIR D_i^+ which is along the direction defined by $e_x^{(k)}$. These symbols arise in the notation for the one-dimensional stability spaces for sets of subgroups which are denoted by a single symbol, e.g. $C_{2h}^{(xj)}$, $j = 1, 2, \ldots, 5$. The value of the superscript 'k' depends on both the value of the index 'j' of the set of subgroups and on the value of the subscript 'i' of the stability space. The values of $k = k(i, j)$ are given in table 11. For example, the value of the superscript k in the symbol for the stability space $E_{5x}^{+(k)}$ of the point group $C_{2h}^{(x2)}$ is $k = 3$ since $i = 5$ and $j = 2$.

Central to the application of group theoretical criteria (Birman 1966, Goldrich and Birman 1968, Jaric and Birman 1977, Jaric 1981, 1982) to determine the possible symmetries which can arise via a continuous phase transition is the calculation of subduction frequencies. Subduction frequencies are the number of times the identity representation is contained in the PIR D_j of G_0 subduced onto a subgroup G of G_0. The subduction frequency of a PIR D_j of G_0 with respect to the subgroup G is equal to the dimension of the stability space of D_j with respect to G. Consequently, the subduction frequencies of the PIR of the point groups C_{10h} and D_{10h} can be found from tables 9 and 10 where the stability spaces of the PIR of the point groups C_{10h} and D_{10h} are respectively given.

For each PIR of C_{10h} and D_{10h} we list in table 12 those subgroups, called epikernels, which satisfy the chain subduction criterion. These are the possible symmetries which can arise via a phase transition where the transition order parameters transform as basis functions of the corresponding PIR. Among the subgroups which satisfy the chain

Table 8. Elements of the point group D_{10h} and its subgroups. The index j takes the values $j = 1, 2, \ldots, 5$.

$D_{10h}(10_z/m_zm_xm_y)$	1	10_z	5_z	10_z^3	5_z^2	2_z	5_z^{-2}	10_z^{-3}	5_z^{-1}	10_z^{-1}
	$2_x^{(1)}$	$2_y^{(5)}$	$2_x^{(4)}$	$2_y^{(3)}$	$2_x^{(2)}$	$2_y^{(1)}$	$2_x^{(5)}$	$2_y^{(4)}$	$2_x^{(3)}$	$2_y^{(2)}$
	$\bar{1}$	$\overline{10}_z$	$\bar{5}_z$	$\overline{10}_z^3$	$\bar{5}_z^{-3}$	m_z	$\bar{5}_z^3$	$\overline{10}_z^{-3}$	$\bar{5}_z^{-1}$	$\overline{10}_z^{-1}$
	$m_x^{(1)}$	$m_y^{(5)}$	$m_x^{(4)}$	$m_y^{(3)}$	$m_x^{(2)}$	$m_y^{(1)}$	$m_x^{(5)}$	$m_y^{(4)}$	$m_x^{(3)}$	$m_y^{(2)}$
$D_{10}(10_z2_x2_y)$	1	10_z	5_z	10_z^3	5_z^2	2_z	5_z^{-2}	10_z^{-3}	5_z^{-1}	10_z^{-1}
	$2_x^{(1)}$	$2_y^{(5)}$	$2_x^{(4)}$	$2_y^{(3)}$	$2_x^{(2)}$	$2_y^{(1)}$	$2_x^{(5)}$	$2_y^{(4)}$	$2_x^{(3)}$	$2_y^{(2)}$
$D_{5d}^x(\bar{5}_z2_x/m_x)$	1	$\bar{5}_z$	5_z^2	$\bar{5}_z^3$	5_z^{-1}	$\bar{1}$	5_z	$\bar{5}_z^{-3}$	5_z^{-2}	$\bar{5}_x^{-1}$
	$2_x^{(1)}$	$m_x^{(4)}$	$2_x^{(2)}$	$m_x^{(5)}$	$2_x^{(3)}$	$m_x^{(1)}$	$2_x^{(4)}$	$m_x^{(2)}$	$2_x^{(5)}$	$m_x^{(3)}$
$D_{5d}^y(\bar{5}_z2_y/m_y)$	1	$\bar{5}_z$	5_z^2	$\bar{5}_z^3$	5_z^{-1}	$\bar{1}$	5_z	$\bar{5}_z^{-3}$	5_z^{-2}	$\bar{5}_x^{-1}$
	$2_y^{(1)}$	$m_y^{(4)}$	$2_y^{(2)}$	$m_y^{(5)}$	$2_y^{(3)}$	$m_y^{(1)}$	$2_y^{(4)}$	$m_y^{(2)}$	$2_y^{(5)}$	$m_y^{(3)}$
$C_{10h}(10_z/m_z)$	1	10_z	5_z	10_z^3	5_z^2	2_z	5_z^{-2}	10_z^{-3}	5_z^{-1}	10_z^{-1}
	$\bar{1}$	$\overline{10}_z$	$\bar{5}_z$	$\overline{10}_z^3$	$\bar{5}_z^{-3}$	m_z	$\bar{5}_z^3$	$\overline{10}_z^{-3}$	$\bar{5}_z^{-1}$	$\overline{10}_z^{-1}$
$D_{5h}^x(\overline{10}_z2_xm_y)$	1	$\overline{10}_z$	5_z	$\overline{10}_z^3$	5_z^2	m_z	5_z^3	$\overline{10}_z^{-3}$	5_z^{-1}	$\overline{10}_z^{-1}$
	$2_x^{(1)}$	$m_y^{(5)}$	$2_x^{(4)}$	$m_y^{(3)}$	$2_x^{(2)}$	$m_y^{(1)}$	$2_x^{(5)}$	$m_y^{(4)}$	$2_x^{(3)}$	$m_y^{(2)}$
$D_{5h}^y(\overline{10}_zm_x2_y)$	1	$\overline{10}_z$	5_z	$\overline{10}_z^3$	5_z^2	m_z	5_z^3	$\overline{10}_z^{-3}$	5_z^{-1}	$\overline{10}_z^{-1}$
	$m_x^{(1)}$	$2_y^{(5)}$	$m_x^{(4)}$	$2_y^{(3)}$	$m_x^{(2)}$	$2_y^{(1)}$	$m_x^{(5)}$	$2_y^{(4)}$	$m_x^{(3)}$	$2_y^{(2)}$
$C_{10v}(10_zm_xm_y)$	1	10_z	5_z	10_z^3	5_z^2	2_z	5_z^{-2}	10_z^{-3}	5_z^{-1}	10_z^{-1}
	$m_x^{(1)}$	$m_y^{(5)}$	$m_x^{(4)}$	$m_y^{(3)}$	$m_x^{(2)}$	$m_y^{(1)}$	$m_x^{(5)}$	$m_y^{(4)}$	$m_x^{(3)}$	$m_y^{(2)}$
$C_{5i}(\bar{5}_z)$	1	$\bar{5}_z$	5_z^2	$\bar{5}_z^3$	5_z^{-1}	$\bar{1}$	5_z	$\bar{5}_z^{-3}$	5_z^{-2}	$\bar{5}_z^{-1}$
$D_5^x(5_z2_x1)$	1	5_z	5_z^2	5_z^{-2}	5_z^{-1}	$2_x^{(1)}$	$2_x^{(4)}$	$2_x^{(2)}$	$2_x^{(5)}$	$2_x^{(3)}$
$D_5^y(5_z12_y)$	1	5_z	5_z^2	5_z^{-2}	5_z^{-1}	$2_y^{(1)}$	$2_y^{(4)}$	$2_y^{(2)}$	$2_y^{(5)}$	$2_y^{(3)}$
$C_{10}(10_z)$	1	10_z	5_z	10_z^3	5_z^2	2_z	5_z^{-2}	10_z^{-3}	5_z^{-1}	10_z^{-1}
$C_{5v}^x(5_zm_x1)$	1	5_z	5_z^2	5_z^{-2}	5_z^{-1}	$m_x^{(1)}$	$m_x^{(4)}$	$m_x^{(2)}$	$m_x^{(5)}$	$m_x^{(3)}$
$C_{5v}^y(5_z1m_y)$	1	5_z	5_z^2	5_z^{-2}	5_z^{-1}	$m_y^{(1)}$	$m_y^{(4)}$	$m_y^{(2)}$	$m_y^{(5)}$	$m_y^{(3)}$
$C_{5h}(\overline{10}_z)$	1	$\overline{10}_z$	5_z	$\overline{10}_z^3$	5_z^2	$\bar{1}$	5_z^{-2}	$\overline{10}_2^{-3}$	5_z^{-1}	$\overline{10}_z^{-1}$
$C_5(5_z)$	1	5_z	5_z^2	5_z^{-2}	5_z^{-1}					
$D_{2h}^{(j)}(m_zm_x^{(j)}m_y^{(j)})$	1	2_z	$2_x^{(j)}$	$2_y^{(j)}$	$\bar{1}$	m_z	$m_x^{(j)}$	$m_y^{(j)}$		
$D_2^{(j)}(2_z2_x^{(j)}2_y^{(j)})$	1	2_z	$2_x^{(j)}$	$2_y^{(j)}$						
$C_{2h}^{(xj)}(2_x^{(j)}/m_x^{(j)})$	1	$2_x^{(j)}$	$\bar{1}$	$m_x^{(j)}$						
$C_{2h}^{(yj)}(2_y^{(j)}/m_y^{(j)})$	1	$2_y^{(j)}$	$\bar{1}$	$m_y^{(j)}$						
$C_{2h}(2_z/m_z)$	1	2_z	$\bar{1}$	m_z						
$C_{2v}^{(xj)}(m_z2_x^{(j)}m_y^{(j)})$	1	$2_x^{(j)}$	m_z	$m_y^{(j)}$						
$C_{2v}^{(yj)}(m_zm_x^{(j)}2_y^{(j)})$	1	$2_y^{(j)}$	m_z	$m_x^{(j)}$						
$C_{2v}^{(j)}(2_zm_x^{(j)}m_y^{(j)})$	1	2_z	$m_x^{(j)}$	$m_y^{(j)}$						
$C_i(\bar{1})$	1	$\bar{1}$								
$C_{2x}^{(j)}(2_x^{(j)})$	1	$2_x^{(j)}$								
$C_{2y}^{(j)}(2_y^{(j)})$	1	$2_y^{(j)}$								
$C_2(2_z)$	1	2_z								
$C_s^{(xj)}(m_x^{(j)})$	1	$m_x^{(j)}$								
$C_s^{(yj)}(m_y^{(j)})$	1	$m_y^{(j)}$								
$C_s(m_z)$	1	m_z								
$C_1(1)$	1									

subduction criterion listed in table 12, we have underlined for each PIR that subgroup which satisfies the kernel-core criterion (Ascher 1977, Kopsky 1980, 1982, Litvin *et al* 1982). These subgroups are the kernels of the corresponding PIR. We have also determined that all PIR of C_{10h} and D_{10h}, except, of course, the identity representation, satisfy the Landau stability criterion and all PIR satisfy the Lifshitz homogeneity criterion for phase transitions (Landau and Lifshitz 1958).

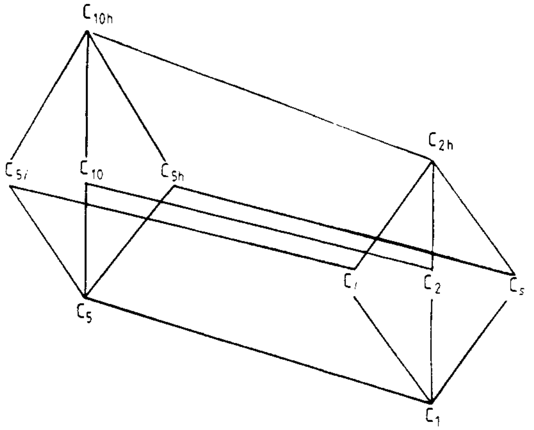

Figure 2. The lattice of subgroups of C_{10h}.

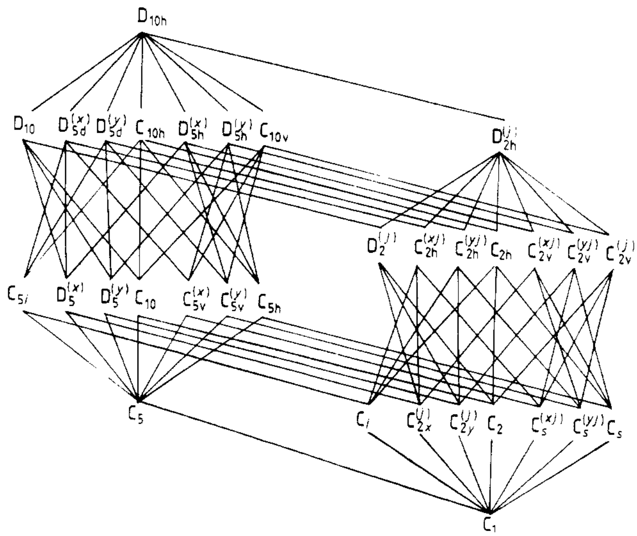

Figure 3. The lattice of subgroups of D_{10h}.

Table 9. Stability spaces of the PIR of the point group C_{10h}.

		C_{5i}	X_1^+, X_2^+		
C_{10h}	X_1^+	C_{10}	X_1^+, X_1^-	C_5	$X_1^+, X_2^+, X_1^-, X_2^-$
		C_{5h}	X_1^+, X_2^-		
		C_i	$X_1^+, X_2^+, D_5^+, D_6^+, D_7^+, D_8^+$	C_1	$X_1^+, X_2^+, X_1^-, X_2^-, D_5^+, D_6^+,$
C_{2h}	X_1^+, D_5^+, D_6^+	C_2	$X_1^+, X_1^-, D_5^+, D_6^+, D_5^-, D_6^-$		$D_7^+, D_8^+, D_5^-, D_6^-, D_7^-, D_8^-$
		C_s	$X_1^+, X_2^-, D_5^+, D_6^+, D_7^-, D_8^-$		

Table 10. Stability spaces of the PIR of the point group D_{10h}.

		D_{10}	X_1^+, X_1^-	C_{5i}	$X_1^+, X_2^+, X_3^+, X_4^+$	C_i	$X_1^+, X_2^+, X_3^+, X_4^+, D_5^+, D_6^+, D_7^+, D_8^+$
		$D_{5d}^{(x)}$	X_1^+, X_3^+	$D_5^{(x)}$	$X_1^+, X_3^+, X_1^-, X_3^-$	$C_{2x}^{(z)}$	$X_1^+, X_1^-, X_3^+, X_3^-, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{7x}^{+(k)}, E_{8x}^{+(k)}, E_{5x}^{-(k)}, E_{6x}^{-(k)}, E_{7x}^{-(k)}, E_{8x}^{-(k)}$
		$D_{5d}^{(y)}$	X_1^+, X_4^+	$D_5^{(y)}$	$X_1^+, X_4^+, X_1^-, X_4^-$	$C_{2y}^{(z)}$	$X_1^+, X_1^-, X_4^+, X_4^-, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{7y}^{+(k)}, E_{8y}^{+(k)}, E_{5x}^{-(k)}, E_{6x}^{-(k)}, E_{7y}^{-(k)}, E_{8y}^{-(k)}$
D_{10h}	X_1^+	C_{10h}	X_1^+, X_2^+	C_{10}	$X_1^+, X_2^+, X_1^-, X_2^-$	C_2	$X_1^+, X_2^+, X_1^-, X_2^-, D_5^+, D_6^+, D_5^-, D_6^-$
		$D_{5h}^{(x)}$	X_1^+, X_3^-	$C_{5v}^{(x)}$	$X_1^+, X_3^+, X_2^-, X_4^-$	$C_s^{(xz)}$	$X_1^+, X_3^+, X_2^-, X_4^-, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{7x}^{+(k)}, E_{8y}^{+(k)}, E_{5y}^{-(k)}, E_{6y}^{-(k)}, E_{7y}^{-(k)}, E_{8y}^{-(k)}$
		$D_{5h}^{(y)}$	X_1^+, X_4^-	$C_{5v}^{(y)}$	$X_1^+, X_4^+, X_2^-, X_3^-$	$C_s^{(yz)}$	$X_1^+, X_4^+, X_2^-, X_3^-, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{7y}^{+(k)}, E_{8y}^{+(k)}, E_{5y}^{-(k)}, E_{6y}^{-(k)}, E_{7x}^{-(k)}, E_{8x}^{-(k)}$
		C_{10v}	X_1^+, X_2^-	C_{5h}	$X_1^+, X_2^+, X_3^-, X_4^-$	C_s	$X_1^+, X_2^+, X_3^-, X_4^-, D_5^+, D_6^+, D_7^-, D_8^-$
			$D_2^{(z)}$	$X_1^+, X_1^-, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{5x}^{-(k)}, E_{6x}^{-(k)}$			
			$C_{2h}^{(xz)}$	$X_1^+, X_3^+, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{7x}^{+(k)}, E_{8x}^{+(k)}$			
			$C_{2h}^{(yz)}$	$X_1^+, X_4^+, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{7y}^{+(k)}, E_{8y}^{+(k)}$		C_5	$X_1^+, X_2^+, X_3^+, X_4^+, X_1^-, X_2^-, X_3^-, X_4^-$
$D_{2h}^{(z)}$	$X_1^+, E_{5x}^{+(k)}, E_{6x}^{+(k)}$		C_{2h}	$X_1^+, X_2^+, D_5^+, D_6^+$			
			$C_{2v}^{(xz)}$	$X_1^+, X_3^-, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{7x}^{-(k)}, E_{8x}^{-(k)}$		C_1	$X_1^+, X_2^+, X_3^+, X_4^+, X_1^-, X_2^-, X_3^-, X_4^-, D_5^+, D_6^+, D_7^+, D_8^+, D_5^-, D_6^-, D_7^-, D_8^-$
			$C_{2v}^{(yz)}$	$X_1^+, X_4^-, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{7y}^{-(k)}, E_{8y}^{-(k)}$			
			$C_{2x}^{(z)}$	$X_1^+, X_2^-, E_{5x}^{+(k)}, E_{6x}^{+(k)}, E_{5y}^{-(k)}, E_{6y}^{-(k)}$			

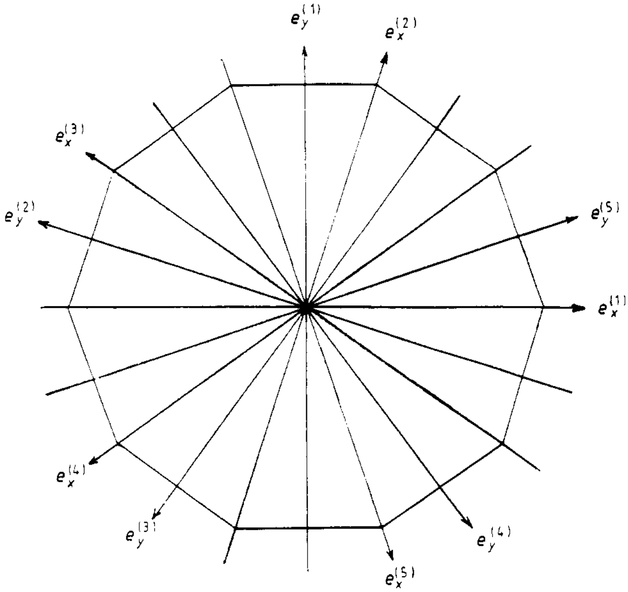

Figure 4. Directions of vectors $e_x^{(j)}$ and $e_y^{(j)}$, $j = 1, 2, \ldots, 5$, in the two-dimensional space spanned by the basis functions of two-dimensional PIR.

Table 11. The value of the index $k = k(i,j)$ for specific values of the indices i and j is given at the intersection of the ith row and jth column.

	j				
i	1	2	3	4	5
5	1	3	5	2	4
6	1	5	4	3	2
7	1	2	3	4	5
8	1	4	2	5	3

4. Tensorial covariants

Tensorial covariants are linear combinations of components of a tensor which transform as basis functions of irreducible representations of a group. We derive here tensorial covariants for a wide variety of tensors and the PIR of the point groups C_{10h} and D_{10h}. In table 13 we list the tensors which we consider, their parity, intrinsic symmetry in Jahn (1949) notation and examples of corresponding physical tensors. We shall use the following conventional abbreviated notation for the components of symmetric second-rank tensors u_{ij}:

$$u_1 = u_{xx} \qquad u_2 = u_{yy} \qquad u_3 = u_{zz}$$
$$u_4 = 2u_{yz} \qquad u_5 = 2u_{zx} \qquad u_6 = 2u_{xy}.$$

The tensor covariants are derived using the tables of Clebsch–Gordan products. This is the same method which has been applied to obtain the tensorial covariants of the magnetic and non-magnetic cyrstallographic point groups (Kopsky 1979b). The tensorial covariants for the tensors given in table 13 for the point groups D_{10h} and C_{10h} are given, respectively, in tables 14 and 15.

Table 12. For each PIR of the point groups C_{10h} and D_{10h} we list those subgroups which satisfy the chain subduction criterion. The epikernel of each PIR is underlined.

D_{10h}		C_{10h}	
1+	$\underline{D_{10h}}$	1+	$\underline{C_{10h}}$
2+	$\underline{C_{10h}}$	2+	$\underline{C_{5i}}$
3+	$\underline{D_{5d}^{(x)}}$		
4+	$\underline{D_{5d}^{(y)}}$		
5+	$D_{2h}^{(z)}, \underline{C_{2h}}$	5+	$\underline{C_{2h}}$
6+	$D_{2h}^{(z)}, \underline{C_{2h}}$	6+	$\underline{C_{2h}}$
7+	$C_{2h}^{(xz)}, C_{2h}^{(yz)}, \underline{C_i}$	7+	$\underline{C_i}$
8+	$C_{2h}^{(xz)}, C_{2h}^{(yz)}, \underline{C_i}$	8+	$\underline{C_i}$
1−	$\underline{D_{10}}$	1−	$\underline{C_{10}}$
2−	$\underline{C_{10v}}$	2−	$\underline{C_{5h}}$
3−	$\underline{D_{5h}^{(x)}}$		
4−	$\underline{D_{5h}^{(y)}}$		
5−	$C_{2v}^{(z)}, D_2^{(z)}, \underline{C_2}$	5−	$\underline{C_2}$
6−	$C_{2v}^{(z)}, D_2^{(z)}, \underline{C_2}$	6−	$\underline{C_2}$
7−	$C_{2v}^{(xz)}, C_{2v}^{(yz)}, \underline{C_s}$	7−	$\underline{C_s}$
8−	$C_{2v}^{(xz)}, C_{2v}^{(yz)}, \underline{C_s}$	8−	$\underline{C_s}$

Table 13. List of tabulated tensors.

Tensor	Parity	Jahn symbol	Physical tensor
ε	−		Pseudoscalar, enanthiomorphism
P	−	V	Polarisation
u	+	$[V^2]$	Strain, stress, permittivity
d	−	$V[V^2]$	Piezoelectric tensor, electro-optic coefficient
s	+	$[[V^2]^2]$	Electric compliance or stiffness coefficient
Q	+	$[V^2]^2$	Electrostriction, elasto-optic or piezo-optic tensor
g	−	$[V^2]$	Gyration tensor or optical rotary power
A	+	$V[V^2]$	Electrogyration tensor
<u>Relations</u>			
$u \sim [P \otimes P]$			$Q = Q^{sym} + Q^{anti}$
$d \sim P \otimes u$			$Q^{sym} = \frac{1}{2}(Q_{ij} + Q_{ji}) = s_{ij}$
$s \sim [u \otimes u]$			$Q^{anti} = \frac{1}{2}(Q_{ij} - Q_{ji}) = q_{ij}$
$Q \sim u \otimes u$			
$g \sim u$			
$A \sim d$			

The properties of a physical system in equilibrium must be invariant under the operations of its symmetry group, while the non-invariant properties must vanish. The invariant combinations of tensor components are given in the column under D_1^+ in both tables 14 and 15. Equating all other covariants to zero, one obtains a set of conditions which the equilibrium tensor components must satisfy. These conditions are given in brackets in the D_1^+ column of both tables 14 and 15.

There are only two types of tensors among those listed where the equilibrium form can be used to distinguish between the point groups D_{10h} and C_{10h}. These are the tensors denoted by A and by $q = Q^{anti}$.

Table 14. Tensorial covariants for the point group D_{10h}.

$D_1^+(X_1^+)$	$D_2^+(X_2^+)$	$D_3^+(X_3^+)$	$D_4^+(X_4^+)$	$D_5^+(X_5^+, Y_5^+)$	$D_6^+(X_6^+, Y_6^+)$	$D_7^+(X_7^+, Y_7^+)$	$D_8^+(X_8^+, Y_8^+)$
u_1+u_2, u_3	$q_{16}-q_{26}, q_{45}$			(u_1-u_2, u_6)	$(s_{11}+s_{22}-2s_{12}-s_{66},$	$(u_4, -u_5)$	$(s_{14}-s_{24}+s_{56},$
$[u_1=u_2]$	$A_{15}+A_{24}, A_{33}$			$(s_{11}-s_{22}, s_{16}+s_{26})$	$2s_{16}-2s_{26})$	$(s_{14}+s_{24}, -s_{15}-s_{25})$	$s_{25}-s_{15}+s_{46})$
$s_{11}+s_{22}+2s_{12}$	$A_{31}+A_{32}$			$(s_{13}-s_{23}, s_{36})$		$(s_{14}-s_{24}-s_{56},$	$(q_{14}-q_{24}-q_{56},$
$s_{11}+s_{22}-2s_{12}+s_{66}$				$(s_{44}-s_{55}, -2s_{45})$		$s_{15}-s_{25}+s_{46})$	$q_{25}-q_{15}-q_{46})$
$s_{13}+s_{23}, s_{33}$				$(-2q_{12}, q_{16}+q_{26})$		$(s_{34}, -s_{35})$	$(A_{11}-A_{12}-A_{26},$
$s_{44}+s_{55}$				$(q_{31}-q_{32}, q_{36})$		$(q_{14}+q_{24}, -q_{15}-q_{25})$	$A_{21}-A_{22}+A_{16})$
$[s_{11}=s_{22}=s_{12}+s_{66}/2,$				$(A_{14}+A_{25}, A_{24}-A_{15})$		$(q_{34}, -q_{35})$	
$s_{13}=s_{23}, s_{44}=s_{55}]$				$(A_{36}, A_{32}-A_{31})$		$(q_{14}-q_{24}+q_{56},$	
$q_{13}+q_{23}$						$q_{15}-q_{25}-q_{46})$	
$[q_{13}=q_{23}]$						(A_{13}, A_{23})	
$A_{14}-A_{25}$						(A_{35}, A_{34})	
$[A_{14}=-A_{25}]$						$(A_{11}+A_{12}, A_{21}+A_{22})$	
						$(A_{11}-A_{12}+A_{26},$	
						$A_{22}-A_{21}+A_{16})$	
$D_1^-(X_1^-)$	$D_2^-(X_2^-)$	$D_3^-(X_3^-)$	$D_4^-(X_4^-)$	$D_5^-(X_5^-, Y_5^-)$	$D_6^-(X_6^-, Y_6^-)$	$D_7^-(X_7^-, Y_7^-)$	$D_8^-(X_8^-, Y_8^-)$
ε	P_3			(g_1-g_2, g_6)		(P_1, P_2)	$(d_{11}-d_{12}-d_{26},$
g_1+g_2, g_3	$d_{15}+d_{24}, d_{33}$			$(d_{14}+d_{25}, d_{24}-d_{15})$		(g_4, g_5)	$d_{21}-d_{22}+d_{16})$
$d_{14}-d_{25}$	$d_{31}+d_{32}$			$(d_{36}, d_{32}-d_{31})$		(d_{13}, d_{23})	
						(d_{35}, d_{34})	
						$(d_{11}-d_{12}+d_{26},$	
						$d_{22}-d_{21}+d_{16})$	
						$(d_{11}+d_{12}, d_{21}+d_{22})$	

Table 15. Tensorial covariants for the point group C_{10h}.

$D_1^+(X_1^+)$	$D_2^+(X_2^+)$	$D_1^-(X_1^-)$	$D_2^-(X_2^-)$
u_1+u_2, u_3		ε	
$[u_1=u_2]$		g_1+g_2, g_3	
$s_{11}+s_{22}+2s_{12}$		P_3	
$s_{11}+s_{22}-2s_{12}+s_{66}$		$d_{14}-d_{25}, d_{33}$	
$s_{13}+s_{23}, s_{33}$		$d_{15}+d_{24}$	
$s_{44}+s_{55}$		$d_{31}+d_{32}$	
$[s_{11}=s_{22}=s_{12}+s_{66}/2,$			
$s_{13}=s_{22}, s_{44}=s_{55}]$			
$q_{13}+q_{23}, q_{45}, q_{16}-q_{26}$			
$[q_{13}=q_{23}, q_{16}=q_{26}]$			
$A_{14}-A_{25}, A_{33}$			
$A_{15}+A_{24}, A_{31}+A_{32}$			
$[A_{14}=-A_{25}, A_{15}=A_{24},$			
$A_{31}=A_{32}]$			

The electrogyration tensor is a physical tensor which transforms as the components of a tensor of type A. Gyration, G, is the magnitude of rotation of the plane of polarisation when a plane-polarised beam moves through a crystal (Nye 1964):

$$G = g_{ij}L_iL_j + A_{k(ij)}E_kL_iL_j \tag{3}$$

where $i, j, k = 1, 2, 3$, E is an electric field and L is the distance transversed through the crystal. The gyration tensor g vanishes for both point groups C_{10h} and D_{10h} and A is called the electrogyration tensor.

For the equilibrium form of the electrogyration tensor invariant under D_{10h}, we have from table 14 that $A_{14}=-A_{25}$. Consequently

$$A_{x(yz)} = A_{x(zy)} = -A_{y(zx)} = -A_{y(xz)}$$

and

$$G = 2A_{x(yz)}(E_xL_yL_z - E_yL_xL_z). \tag{4}$$

From table 15, the equilibrium form of the electrogyration tensor invariant under C_{10h} gives

$$G = 2A_{x(yz)}(E_xL_yL_z - E_yL_xL_z) + 2A_{x(xz)}(E_xL_xL_z - E_yL_yL_z) + A_{z(xx)}(E_zL_x^2 + E_zL_y^2) + A_{z(zz)}E_zL_z^2. \tag{5}$$

Comparing equations (4) and (5) one has that an experimental determination of, for example, the $A_{z(zz)}$ component of the electrogyration tensor can determine which of the two point groups, C_{10h} or D_{10h}, is the point group of the decagonal T-phase quasicrystal studied by Bendersky (1985, 1986).

The electrostriction effect can also be used to distinguish between the point groups C_{10h} and D_{10h}. The relationship between strain ε and electric field E can be written as

$$\varepsilon_{jk} = d_{ijk}E_i + \gamma_{(im)(jk)}E_iE_m \tag{6}$$

where d denotes the piezoelectric effect tensor which vanishes for both the point groups C_{10h} and D_{10h} and γ is the electrostriction effect tensor. The electrostriction tensor $\gamma_{(im)(jk)}$ is symmetric with respect to the interchange of the indices i and m, and also to the interchange of the indices j and k. Consequently the electrostriction tensor transforms as the tensor Q of table 13. A tensor of the type Q can be written as a sum of a symmetrical and an antisymmetrical part, i.e. $Q = Q^{\text{sym}} + q$, with $q = Q^{\text{anti}}$. From table 13 Q^{sym} transforms as a tensor s, and from tables 14 and 15 one finds that the equilibrium form of the tensor s is the same for both point groups C_{10h} and D_{10h}. It is the antisymmetrical part $q = Q^{\text{anti}}$ which can distinguish between these two point groups.

The equilibrium form of the electrostriction tensor invariant under the point group D_{10h} is found from the equilibrium form of the tensors s and q in table 14. We obtain the following relationships, equation (6):

$$\begin{aligned}
\varepsilon_{xx} &= \gamma_{(xx)(xx)}E_x^2 + (\gamma_{(xx)(xx)} - \tfrac{1}{2}\gamma_{(xy)(xy)})E_y^2 + \gamma_{(xx)(zz)}E_z^2 \\
\varepsilon_{yy} &= (\gamma_{(xx)(xx)} - \tfrac{1}{2}\gamma_{(xy)(xy)})E_x^2 + \gamma_{(xx)(xx)}E_y^2 + \gamma_{(xx)(zz)}E_z^2 \\
\varepsilon_{zz} &= \gamma_{(zz)(xx)}E_x^2 + \gamma_{(zz)(yy)}E_y^2 + \gamma_{(zz)(zz)}E_z^2 \\
\varepsilon_{yz} &= 2\gamma_{(zy)(yz)}E_yE_z \\
\varepsilon_{zx} &= 2\gamma_{(yz)(yz)}E_xE_z \\
\varepsilon_{xy} &= 2\gamma_{(xy)(xy)}E_xE_y.
\end{aligned} \tag{7}$$

For the point group C_{10h}, the equilibrium form of the electrostriction tensor is found in table 15. The relationships, equation (6), are those given in equation (7) with the following additional terms:

$$\begin{aligned}
\varepsilon_{xx} &= \ldots + (\gamma_{(xx)(xy)} - \gamma_{(xy)(xx)})E_xE_y \\
\varepsilon_{yy} &= \ldots - (\gamma_{(xx)(xy)} - \gamma_{(xy)(xx)})E_xE_y \\
\varepsilon_{zz} &= \ldots \\
\varepsilon_{yz} &= \ldots + (\gamma_{(yz)(zx)} - \gamma_{(zx)(yz)})E_xE_z \\
\varepsilon_{zx} &= \ldots - (\gamma_{(zy)(zx)} - \gamma_{(zx)(yz)})E_yE_z \\
\varepsilon_{xy} &= \ldots - (\gamma_{(xx)(xy)} - \gamma_{(xy)(xx)})E_x^2 - (\gamma_{(xx)(xy)} - \gamma_{(xy)(xx)})E_y^2.
\end{aligned} \tag{8}$$

Experimental determination of any of these additional terms would then determine which of the two point groups, C_{10h} or D_{10h}, is the symmetry group of the decagonal T-phase quasicrystal.

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References

Ascher E 1977 *J. Phys. C: Solid State Phys.* **10** 1365

Bendersky L 1985 *Phys. Rev. Lett.* **55** 1461

—— 1986 *J. Physique* **47** C3 457

Birman J L 1966 *Phys. Rev. Lett.* **17** 1216

Goldrich F E and Birman J L 1968 *Phys. Rev.* **167** 528

Jahn H A 1949 *Acta Crystallogr.* **2** 30

Jaric M V 1981 *Phys. Rev.* B **23** 3460

—— 1982 *Phys. Rev.* B **25** 2015

Jaric M V and Birman J L 1977 *Phys. Rev.* B **16** 2564

Kopsky V 1975 *J. Phys. C: Solid State Phys.* **8** 3251

—— 1976 *J. Phys. C: Solid State Phys.* **9** 3391, 3405

—— 1979a *J. Phys. A: Math. Gen.* **12** 429, 943

—— 1979b *Acta Crystallogr.* A **35** 83, 95

—— 1980 *Ferroelec.* **24** 3

—— 1982 *Group Lattices, Subduction of Bases, and Fine Domain Structure For Magnetic Crystal Point Groups* (Prague: Academia)

—— 1983 *Czech. J. Phys.* B **33** 458, 721, 845

Landau L D and Lifshitz E M 1958 *Statistical Physics* (Oxford: Pergamon)

Litvin D B, Kotzev J N and Birman J L 1982 *Phys. Rev.* B **26** 6947

Nye J N 1964 *Physical Properties of Crystals* (Oxford: Clarendon)

Patera J, Sharp R T and Winternitz P 1978 *J. Math. Phys.* **19** 2362